ASRL TR 133-2

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

AEROELASTIC AND STRUCTURES RESEARCH LABORATORY

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A ROUTINE METHOD FOR THE CALCULATION OF AERODYNAMIC LOADS ON A WING IN THE VICINITY OF INFINITE VORTICES

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SUMMARY

The vortex lattice method for the calculation of loads on a wing advancing in a uniform stream, initiated by Falkner (Reference 1) in 1943 and refined since then by other researchers (References 2,3) was extended to the case of a wing in the vicinity of infinite vortices.

No experimental data or different treatments of the same problem being available in the literature in order to verify the accuracy of the present method, the results in the report had to be limited to checking their convergence with respect to the three parameters involved and investigating what conditions them.

The problem being treated linearly, only the simple case of one infinite vortex at the midspan of a wing at zero angle of attack has been tested, with the understanding that the method can be applied to more complicated cases once its convergence and its validity have been shown.

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NOMENCLATURE

L = location of infinite straight vortex line along wing span, divided by span.

R = vortex core radius divided by span

U = free stream velocity

W = induced velocity at wing surface

X = abcissa along wing span

Y = ordinate along wing chord

Z = distance between wing midsurface and infinite straight vortex

a = $c/2$

c = wing chord

d_x = spacing between two consecutive trailing vortices, divided by span

d_y = spacing between two consecutive bound vortices, divided by chord

m = number of bound vortices

n = number of trailing vortices

Γ = strength of infinite vortex

Γ_{ij} = strength of (element)$_{ij}$ in the vortex lattice

ϕ = angle between infinite straight vortex and free stream

α = angle of attack

$\gamma(Y)$ = vorticity distribution along wing chord

I INTRODUCTION

A thin, rectangular, and untwisted wing of infinite span moving at a constant angle of attack in a uniform stream produces only bound vorticity along its span with no trailing wake behind.

Consider the same idealized wing over an infinite vortex (Fig. 1).

The effect of the votex is to increase the angle of incidence on the upwash side and to decrease it on the downwash side, as well as to increase the magnitude of the velocity on both sides. Since the change in incidence and velocity, and thus in circulation and lift, is not uniform along the span, a trailing vortex sheet must form.

The infinite vortex Γ being free, will not be straight or stationary when interacting with the wing vorticity. It will likely be distorted into an S shape and rotated. To make the problem amenable to a simple solution, the effect of wing vorticity on Γ will be neglected.

The vorticity distribution over the wing has been modified. It is clear that this will also be the case with different wing shapes and different configurations of infinite vortices. (More than one may be present.)

The object of this report is to show how a vortex lattice method can be used to determine the calculation distribution of a wing in the vicinity of infinite vortices.

II DEVELOPMENT OF THE METHOD

Replace the continuous circulation distribution on the wing surface by a set of line vortices which is discrete in the spanwise and chordwise directions. The circulation is constant along a segment parallel to the span, then, after shedding a trailing vortex extending downstream to infinity, takes on another constant value along the next bound vortex segment. The resulting pattern is a "vortex lattice" laid on the wing surface as shown in Fig. 2. It will not be contested that, if the spacing is made infinitesimally small, the correct distribution can be obtained.

Near the wing, the trailing vortices will be neither parallel to the stream nor straight. Because of insufficient knowledge to determine their geometry near the T.E. and for simplicity's sake, it is assumed here that they are infinitely straight and parallel to the stream.

If the wing is thin, the lattice is located on a surface which is the mean of the upper and lower surfaces. For the case of a thick profile, two vortex lattices, one on the upper surface and the other on the lower, may be used (see Reference 2).

Each bound vortex segment with the two trailing vortices emanating from its ends may be regarded as a horseshoe vortex satisfying the condition that vorticity does not stop in the flow (See Fig. 3). The strength of any trailing vortex is equal to the sum of the strengths of the bound vortex segments adjacent to it and upstream from it. Thus, to determine completely the circulation distribution, only the strengths of the bound vortex segments need to be known. This is the next step considered.

At any point, the velocity induced by the vortex lattice, W, is obtained by summing the contribution of each vortex segment (bound or trailing)

$$W = W\left(\Gamma_{11}, \Gamma_{12}, \ldots\ldots, \Gamma_{n(m-1)}, \Gamma_{nm}\right)$$

The Γ_{ij}'s are the strengths of the bound vortex segments to be evaluated.

By satisfying the boundary condition, namely equating at (n x m) "pivotal points" on the wing the total downwash (obtained by summing the contributions of the vortex lattice, the incident infinite vortices, and the normal component of the free stream velocity) to the local slope, (n x m) linear algebraic equations are obtained the solution of which yields the unknown vortex strengths. To solve those (n x m) equations, a digital computer is needed in view of the required dimensions of the lattice for satisfactory accuracy.

III RULES FOR LAYING OUT THE VORTEX LATTICE AND THE PIVOTAL POINTS

The rules established by Rubbert (Reference 2) for laying out the vortex lattice and the pivotal points have been tested first with a wing advancing in a uniform stream, then used throughout for the problem considered in this report.

The rules are:

(1) The line vortices should be equally spaced in the spanwise, as well as the chordwise, directions. If it is desired to bring them closer over certain parts of the wing because of a larger variation in circulation, the spacing should be changed very smoothly using a "cosine law". (This is illustrated in Figure 4.)

(2) If the chordwise spacing between the line vortices is d_y, the first line vortex is located at $d_y/4$ from the L.E. and the last one at $3d_y/4$ from the T.E.

(3) If the spanwise spacing is d_x, the line vortex at the wing tip is located at $d_x/4$ from it.

(4) The pivotal points must be halfway between two consecutive line vortices in the chordwise, as well as the spanwise, directions. The last row of pivotal points is placed at $d_y/4$ from the T.E. (For a theoretical justification of the pivotal point and line vortex locations, see Appendix I.)

IV REMARKS

1) The storage in the computer being limited, it was found that the accuracy hopefully obtained by the use of the cosine law can be achieved by a smaller lattice spacing. For by using a uniforn spacing throughout, simplicity and shortness are gained, and thus, more of the locations in the computer are left for the solution of a larger number of simultaneous equations, permitting a smaller spacing and a better accuracy.

For a constant chord and spanwise spacing, (n x m) was limited to about 150 before overflow occured in the 7094 IBM Computer.

2) It should be pointed out that the above rules are of the utmost importance. In particular, the results are very sensitive to a displacement of the pivotal points from the middle of the lattice cells.

V CASE TESTED AND PRELIMINARY DISCUSSION

A thin, rectangular wing, approximated by a rectangular flat plate, moving at zero agle of attack over an infinite straight line vortex, Γ, which is parallel to the wing plane, at a distance Z below it and which makes an angle ϕ with the free stream was used for the first case.

This case is illustrated in Figure 5 which shows also the notation used.

In plotting the computer solutions,circulation per unit length is first integrated over the chord to get the circulation around the wing. This integration is equivalent to a summation of the strengths of the bound vortex segments at a spanwise station.

The downwash being skew-symmetric over the wing, (see Figure 1) lift and circulation are expected to be also skew-symmetric and, moreover, to vanish at the tips.

Only the most representative solutions are given in this report. Those which are not included corroborate the general results.

For a smooth circulation distribution and a rapid convergence of the solutions two rules, evolved when investigating the variation in d_x, d_y, and L, are observed:-

<u>RULE A:</u> When Z is "small", of the order of 1/2 chord or less, the infinite vortex Γ should be placed on top of a trailing vortex. Otherwise, it may be placed anywhere.

RULE B: In a direction perpendicular to Γ, i.e., that of a nonuniform downwash induced velocity or, in general, of a nonuniform downwash, the lattice spacing should be less than Z. Thus, if $\phi = 0$, d_x is smaller than Z.

Those two rules become of importance only if Γ is very close to the wing, say if Z is smaller than one chord. A difficulty may then arise when laying out, according to the rules of Section III, the lattice vortex over the chosen wing shape. For instance, for the case of a rectangular wing with Γ at an angle ϕ different from zero, the above two rules and the rules of Section III cannot be satisfied simultaneously. A discussion and suggestions are given in Appendix II.

VI CONVERGENCE OF RESULTS

As d_x is made smaller, for a rectangular wing of AR=20, with d_y=.25 chord, ϕ =0, L = .5 and Z = 1 chord, Figure 6 shows four circulation distributions corresponding to four different values of d_x. In Figure 7, the peak value of circulation is seen converging to an asymptotic value as d_x gets smaller. Note that $d_y = 1/m$, $d_x = \frac{2}{2m + 1}$.

As d_y is made smaller for the same case but with d_x fixed at .054, Figure 8 shows that the circulation distribution is practically the same for four different values of d_y. The distribution insensitivity to the chordwise spacing is due to the special case in hand, namely a constant downwash along the chord. Here one bound vortex is obviously sufficient, permitting a smaller d_x. Later, it will be shown that whenever downwash is not constant along the chord, it is necessary to have more than one bound vortex.

As Z is made larger, the circulation distribution should converge to that of the wing in a uniform flow. Figure 9 illustrates the expected convergence of the circulation peak value. Here the asymptotic value, for $Z = \infty$, is zero since the flat plate is at zero angle of attack.

VII SINGULAR CASE: $Z = 0$

Figure 9 shows also that for $Z = 0$, the circulation peak value becomes infinite. This is a consequence of the unrealistic choice of an infinite vortex Γ with no inner core which induces an infinite downwash at the vortex itself. To avoid this singularity, the infinite vortex is assumed to have a finite core rotating as a rigid body inducing a velocity as shown in Figure 10.

No general theory being available to predict the core diameter, several infinite vortices with different cores have been tested. The solutions are plotted in Figure 11. The important feature is that for a core diameter smaller than or equal to d_x, the circulation remains unchanged and the singularity is still present. This is inherent in the vortex lattice method. For Z being zero, Γ is placed on top of a trailing vortex in accordance with Rule A, i.e. halfway between two pivotal points which are separated by a distance d_x; since the core diameter is less than d_x, the linear distribution of downwash in the core, introduced to eliminate the singularity, does not reach the pivotal points.

VIII CENTER OF PRESSURE LOCATION

Once circulation is known, lift is proportional to it and acts through the center of pressure. The center of pressure is determined as follows:

$$Y_{C.P.} = \frac{\int_{Chord} Y\gamma(Y)\,dY}{\int_{Chord} \gamma(Y)\,dY} = \frac{\sum^{m} Y_i \Gamma_i}{\sum^{m} \Gamma_i}$$

where γ (Y) is the circulation distribution along the chord.

It was found that for $\phi = 0$, and all values of L and Z, the center of pressure locus along the span is practically a straight line at the quarter chord. Thus, the infinite vortex did not shift the center of pressure from its location in a uniform flow (see Figure 12). This last result is only a consequence of the case. It is known that, for a flat plate in a uniform flow, the center of pressure is fixed and located at the quarter chord. Since the infinite vortex is parallel to the stream, each spanwise section of width d_x may be regarded as an elementary flat plate in a uniform flow, with constant circulation and a pair of trailing vortices of same strength at its tips. Bearing in mind that the spanwise variation in incidence and velocity due to Γ does not move the center of pressure on each elementary flat plate, it is clear that their locus will still be a straight line at the 1/4 chord.

Now if a chordwise, as well as a spanwise, variation in stream conditions is included, the centers of pressure will be displaced by difference amounts. Such a chordwise variation is introduced by putting the infinite vortex at an angle $\phi \neq 0$ with the stream. Close to Γ , this variation is large and,

thus, the displacement is relatively large. Far from Γ, it is small and the displacement is small. This is illustrated in Figure 13.

With $\phi \neq 0$ and a corresponding change in the center of pressure locus, the chordwise spacing d_y becomes of importance and some convergence is expected as d_y gets smaller. Clearly, if only one bound vortex (at the quarter chord in accordance with the rules for laying out the lattice) is used, the locus is located at the bound vortex itself and the expected C.P. displacement cannot be accounted for; thus, more than one bound vortex is needed. In Figure 14, the peak value in the C.P. locus is plotted versus d_y showing a rapid convergence of the solution as the spacing gets smaller

All the preceding is in agreement with Rule B.

APPENDIX I

1. Location of Pivotal Points

Two different approaches leading to the same result are given:

First Approach (suggested by P. Rubbert):

Consider a pivotal point P on a vortex sheet (in two dimensions). (See Figure 15)

To compute the induced velocity at P, the sheet SS may be divided into equal segments. At the middle of each segment is located a point vortex of strength equal to the average circulation over the segment (as shown in Figure 15).

The replacement of the continuous circulation distribution by discrete point vortices is acceptable for the segments far from P. As for the two segments adjacent to P, consider the following derivation:-

Suppose the segments are small enough so that between 1 and 3 the circulation may be approximated by a linear continuous distribution (see Figure 16). The velocity induced at P due to the vorticity lying between 1 and 3 us given by:

$$W \approx \int_0^{l} \frac{\left\{\left(\frac{d\gamma}{ds}\right)_2 + \left[\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_2\right]\left[1-\frac{s}{l}\right]\right\} ds}{2\pi(l-s)} - \int_{l}^{2l} \frac{\left\{\left(\frac{d\gamma}{ds}\right)_2 + \left[\left(\frac{d\gamma}{ds}\right)_3 - \left(\frac{d\gamma}{ds}\right)_2\right]\left[1-\frac{s}{l}\right]\right\} ds}{2\pi(l-s)}$$

$$\approx \int_0^{l} \frac{\left[\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_2\right] ds}{2\pi l} - \int_{l}^{2l} \frac{\left[\left(\frac{d\gamma}{ds}\right)_3 - \left(\frac{d\gamma}{ds}\right)_2\right] ds}{2\pi l}$$

$$\approx \frac{1}{2\pi}\left[\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_3\right]$$

Now replace segments (1.2) and (2.3) by concentrated vortices Γ_1 and Γ_2 located at their midpoints. Assume these vortices to be of strength equal to the integral of the distributed circulation on the segments.

$$\Gamma_1 = \int_0^l \left\{ \left[\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_2 \right] \left[1 - \frac{s}{l} \right] + \left(\frac{d\gamma}{ds}\right)_2 \right\} ds$$

$$= \left[\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_2 \right] \frac{l}{2} + \left(\frac{d\gamma}{ds}\right)_2 l$$

Similarly, $\Gamma_2 = \left[\left(\frac{d\gamma}{ds}\right)_3 - \left(\frac{d\gamma}{ds}\right)_2 \right] \frac{l}{2} + \left(\frac{d\gamma}{ds}\right)_2 l$

The velocity induced at P by Γ_1 and Γ_2 is:-

$$W = \frac{\Gamma_1}{2\pi l/2} - \frac{\Gamma_2}{2\pi l/2} = \frac{\left(\frac{d\gamma}{ds}\right)_1 - \left(\frac{d\gamma}{ds}\right)_3}{2\pi}$$

which is the same as the answer for a linear continuous distribution.

The foregoing justifies the position of Γ_1 and Γ_2 at the mid points of their segments. Equivalently, P should be placed halfway between Γ_1 and Γ_2. Then, if more than one pivotal point is desired, a correct configuration would be as shown in Figure 17.

Second Approach:

Consider a two dimensional flat plate aerofoil represented by a vortex line of strength Γ at the quarter chord (i.e. at the center of pressure and aerodynamic center). The downwash due to the vortex line equals the normal component $U \sin\alpha$ of the free stream velocity at a distance d from

the quarter chord, given by:- $\frac{\Gamma}{2\pi d} = U\sin\alpha \Rightarrow d = \frac{\Gamma}{2\pi U\sin\alpha}$

but $\Gamma = \pi X = \pi 2a U\sin\alpha$ (see Reference 4, p. 182)

and for a flat plate $2a = c$ (Reference 4, chapt. 6)

then $d = \frac{\pi(cU\sin\alpha)}{2\pi U\sin\alpha} = \frac{c}{2}$

Thus, the downwashes according to the exact theory and the single vortex line representation are equal at the 3/4 chord.

The flat plate may be represented as in Figure 18.

If a lifting surface is represented by the juxtaposition of several flat plates, the configuration of concentrated vortices and pivotal points would be as shown in Figure 19. It turns out that the pivotal points are halfway between two consecutive point vortices (or vice-versa).

It may be noted incidentally that the first point vortex is at c/4 from the L.E. and the last pivotal point at c/4 from the T.E.

APPENDIX II

Alternative lattice geometries will be presented. Although they may not be of practical use (except when the infinite vortex Γ is not parallel to the stream), they would give more insight to the method.

The circulation distribution over the wing is a three-dimensional surface usually varying in all directions. The vortex lattice method is essentially a stepwise calculation of the circulation. Its continuous distribution is replaced by a network of concentrated vortex lines, the strength of each being the average circulation over the corresponding lattice spacing (see Fig. 20).

In the small double-crossed area (Figure 20), a vorticity vector may have an arbitrary direction. However, it may be broken into an X and Y component, each of which is taken into account when evaluating Γ_1 and Γ_2 respectively.

It is important to note that the X and Y directions can be arbitrary; moreover, they need not be perpendicular to each other.

In this report, for the sake of simplicity and shortness, a rectangular lattice with the Y-axis parallel to the free stream was used. As shown in the results, this lattice geometry was adequate for all the cases tested. However, as pointed out after Tables A and B, a difficulty arises in locating the infinite vortex when $\phi \neq 0$ and Γ is small. For that case and others which require non-rectangular lattices, the two lattices suggested below should be adjusted before use. This

is accomplished by testing a simple case with both the rectangular lattice, used in this report, and the chosen lattice; and then adjusting the geometry of the latter to obtain the same results from both.

A way to resolve the difficulty and to satisfy Rules A and B is to use a tilted lattice at an angle ϕ , as illustrated in Figure 21.

The rules for laying out the lattice, (3) for the type A, (2) and (3) for the type B (Figure 21), may not be readily applicable. They would have to be modified when adjusting the lattice geometry. Special care should be given to the wing tips where a few horseshoe vortices would have to be at reduced spacing (Figure 22).

In Figure 22, the location of the pivotal points near the tip is not known à priori. Rule (4) is to be modified at the tips.

An equivalent wing fitting the tilted lattice may probably also be used (Figure 23).

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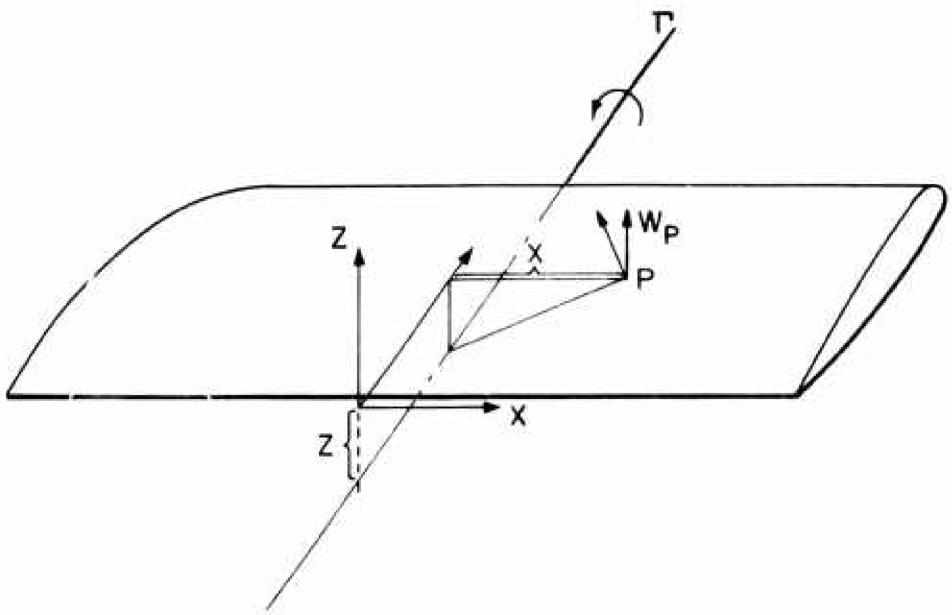
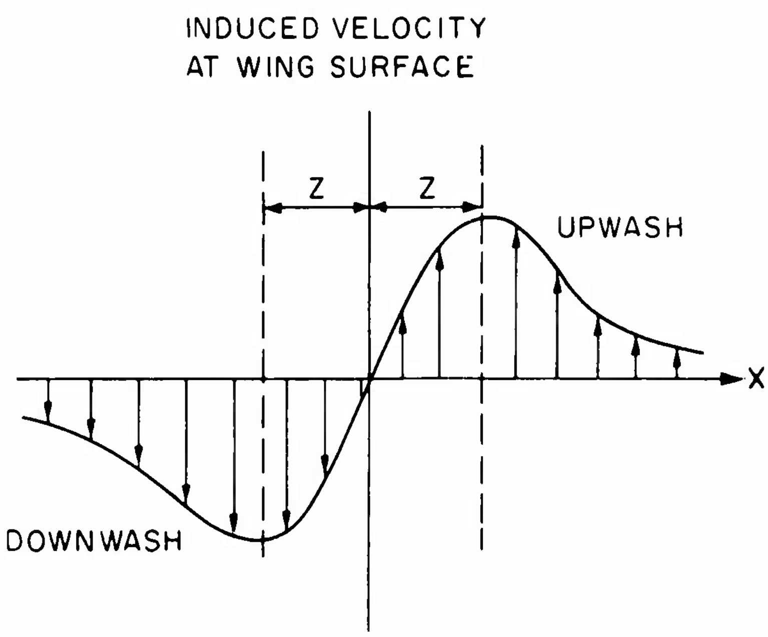




UPWASH INDUCED AT P = W_p =

$$\left[\frac{\Gamma}{2\pi(x^2+z^2)^{1/2}}\right]\left[\frac{x}{(x^2+z^2)^{1/2}}\right] = \frac{\Gamma x}{2\pi(x^2+z^2)}$$

$$\frac{dw_p}{dx} = \frac{\Gamma(Z^2-X^2)}{2\pi(x^2+z^2)} = 0 \text{ @ } X = \pm Z$$

Fig. 1 A Wing Over An Infinite Vortex

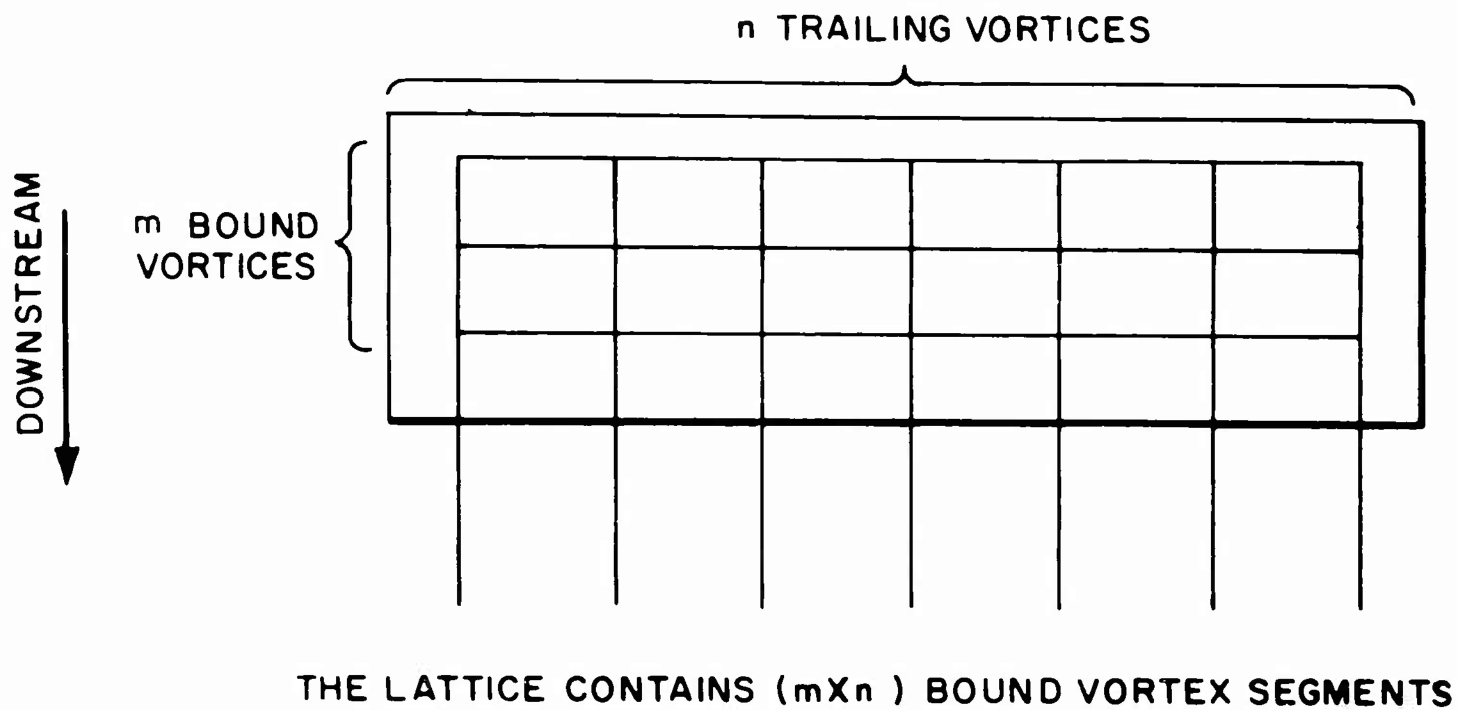


Fig. 2 A Rectangular Vortex Lattice

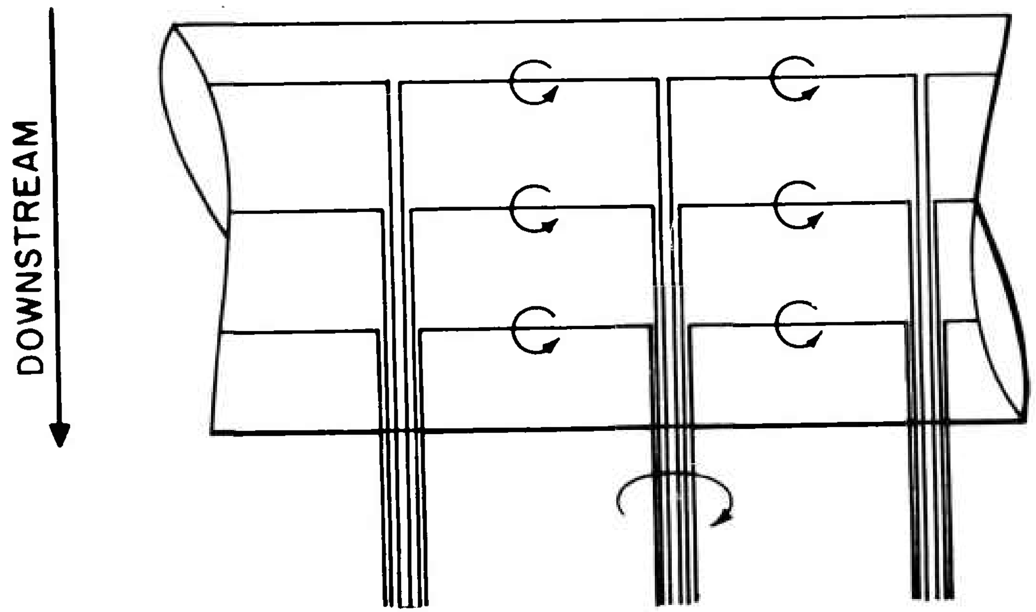


Fig. 3 Detail Of The Vortex Lattice

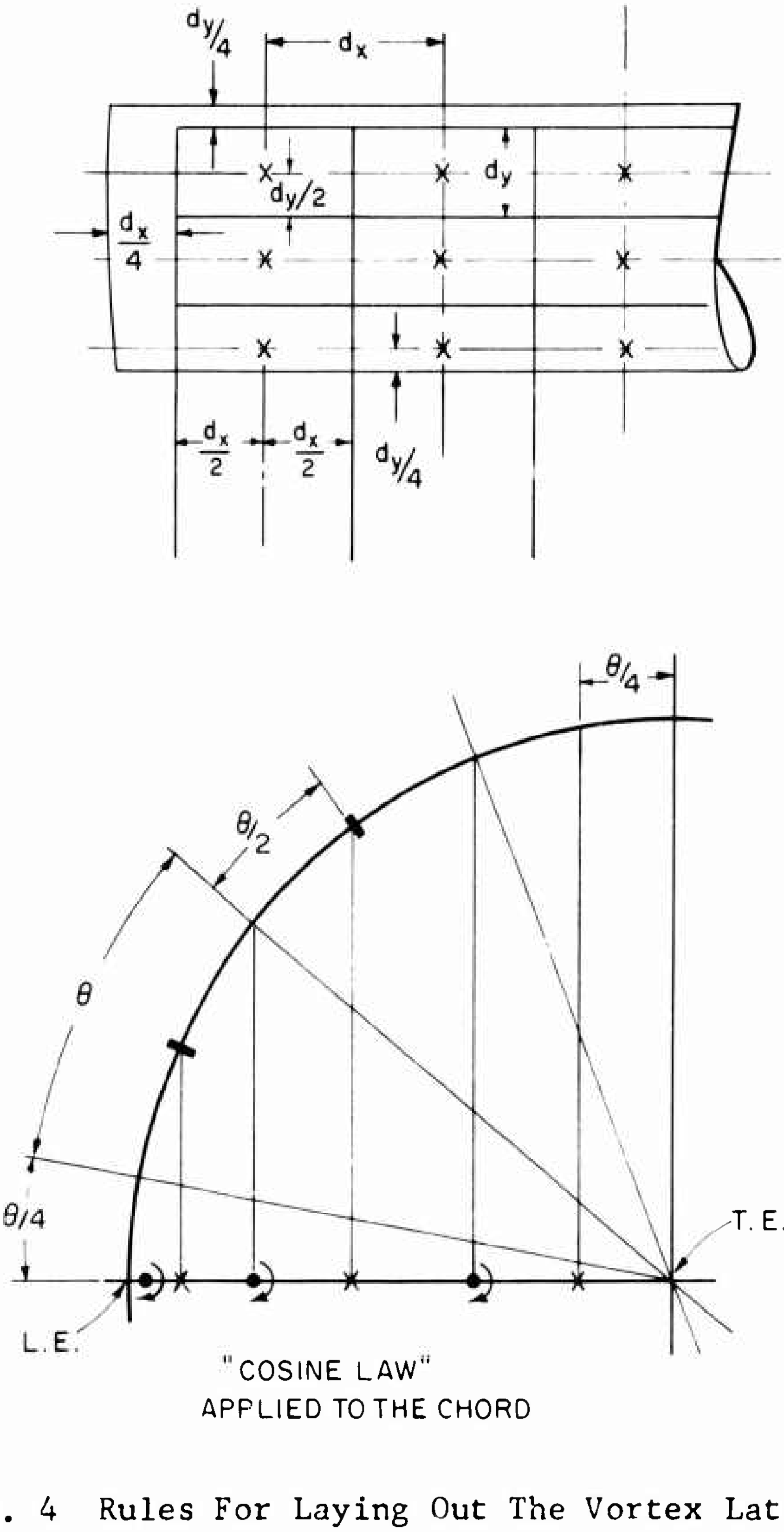


Fig. 4 Rules For Laying Out The Vortex Lattice And The Pivotal Points

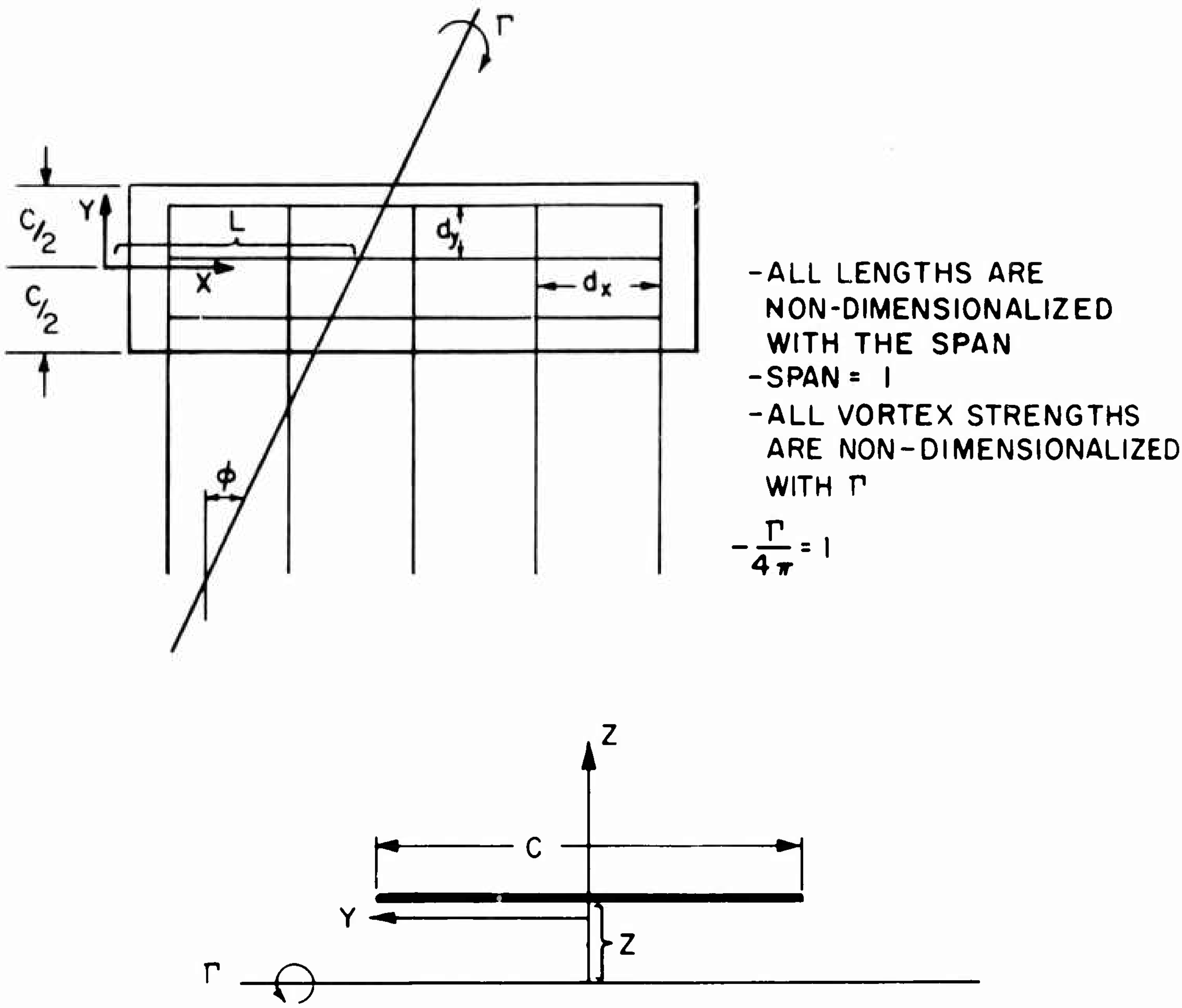


Fig. 5 Notation For The Case Tested In This Report

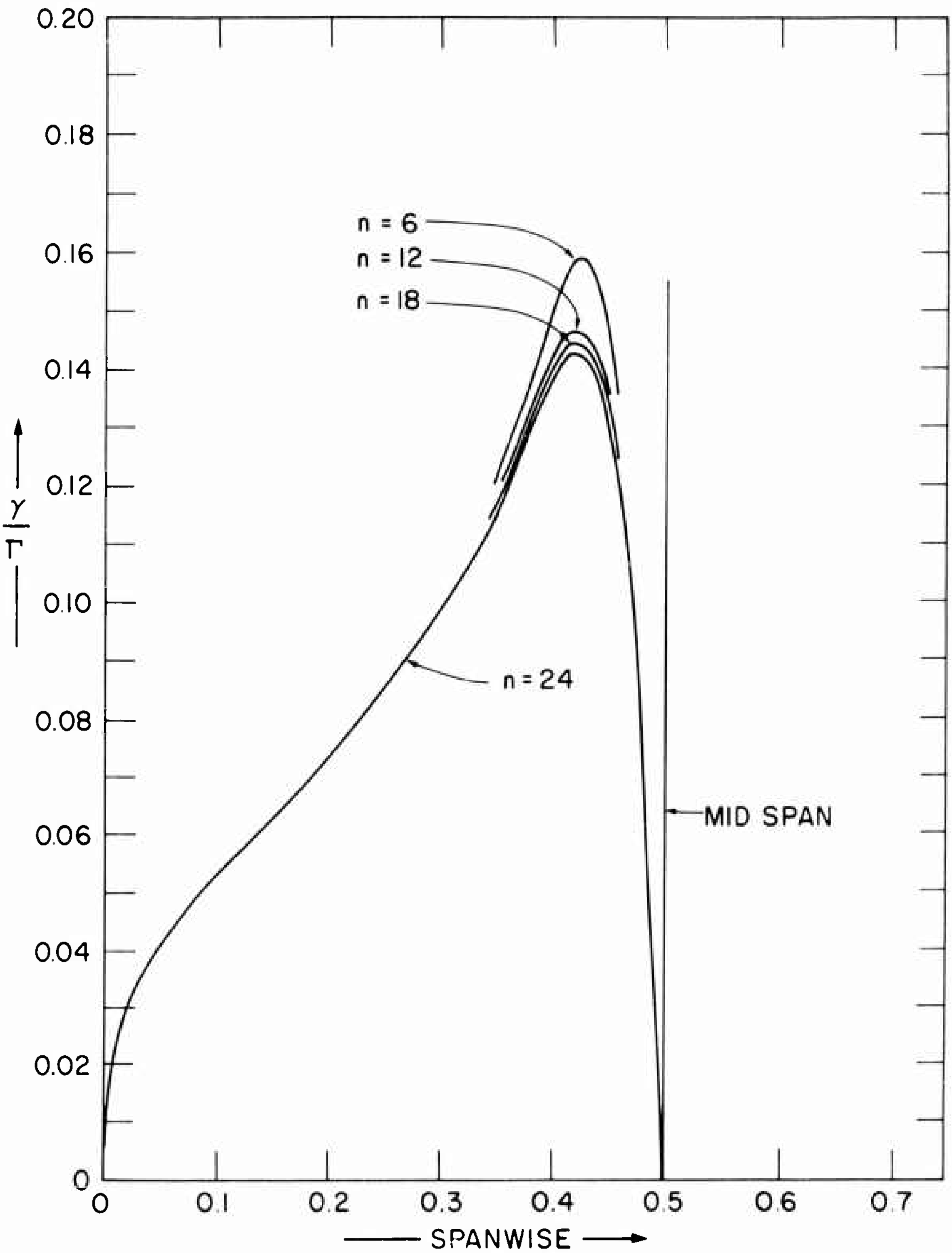


Fig. 6 Spanwise Circulation Distribution For Four Different Values of d_x.
Z=1 chord, AR=20, m=4, ϕ=0, L=0.5

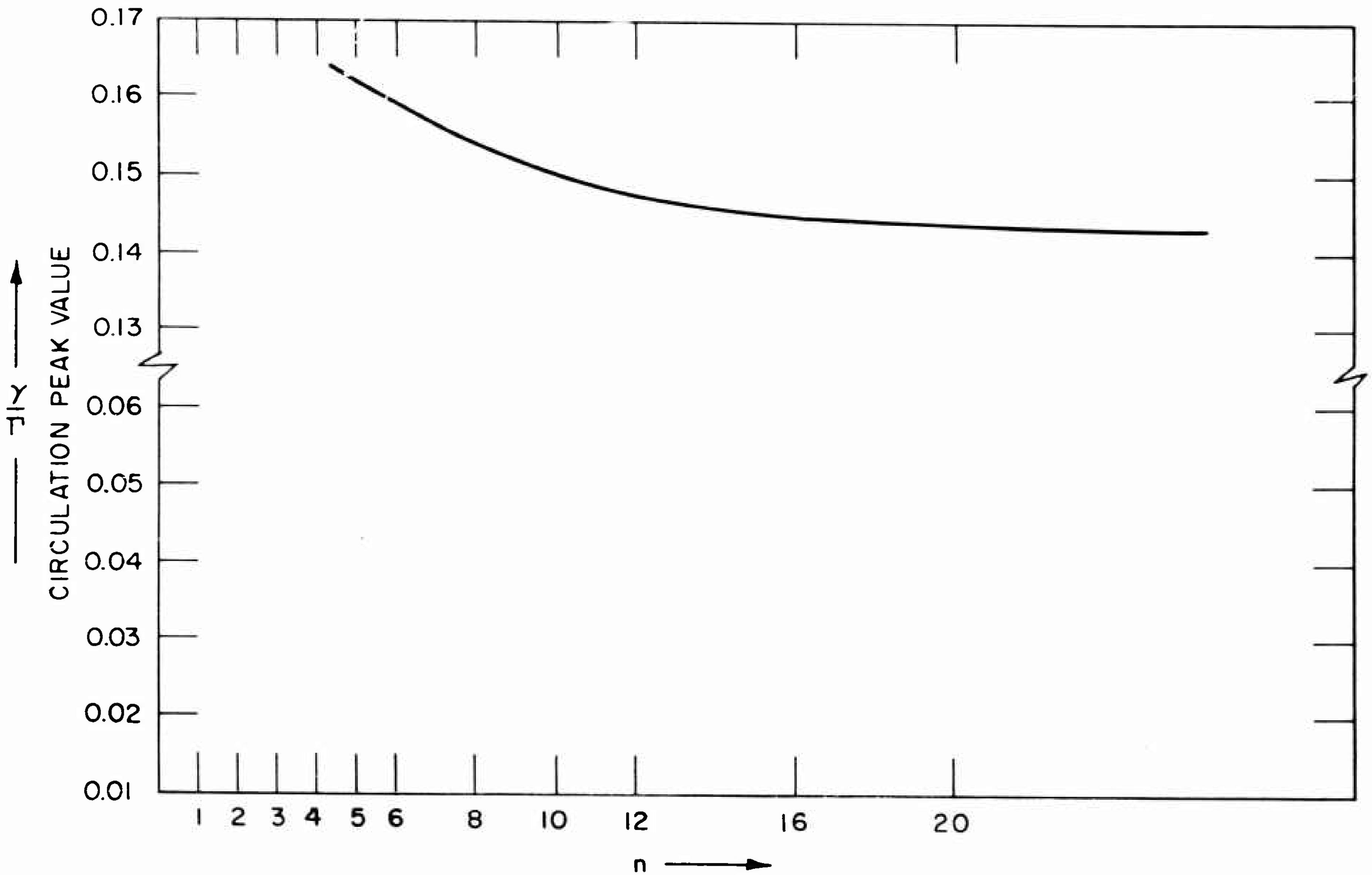


Fig. 7 Convergence Of The Circulation Peak Value As d_x Decreases

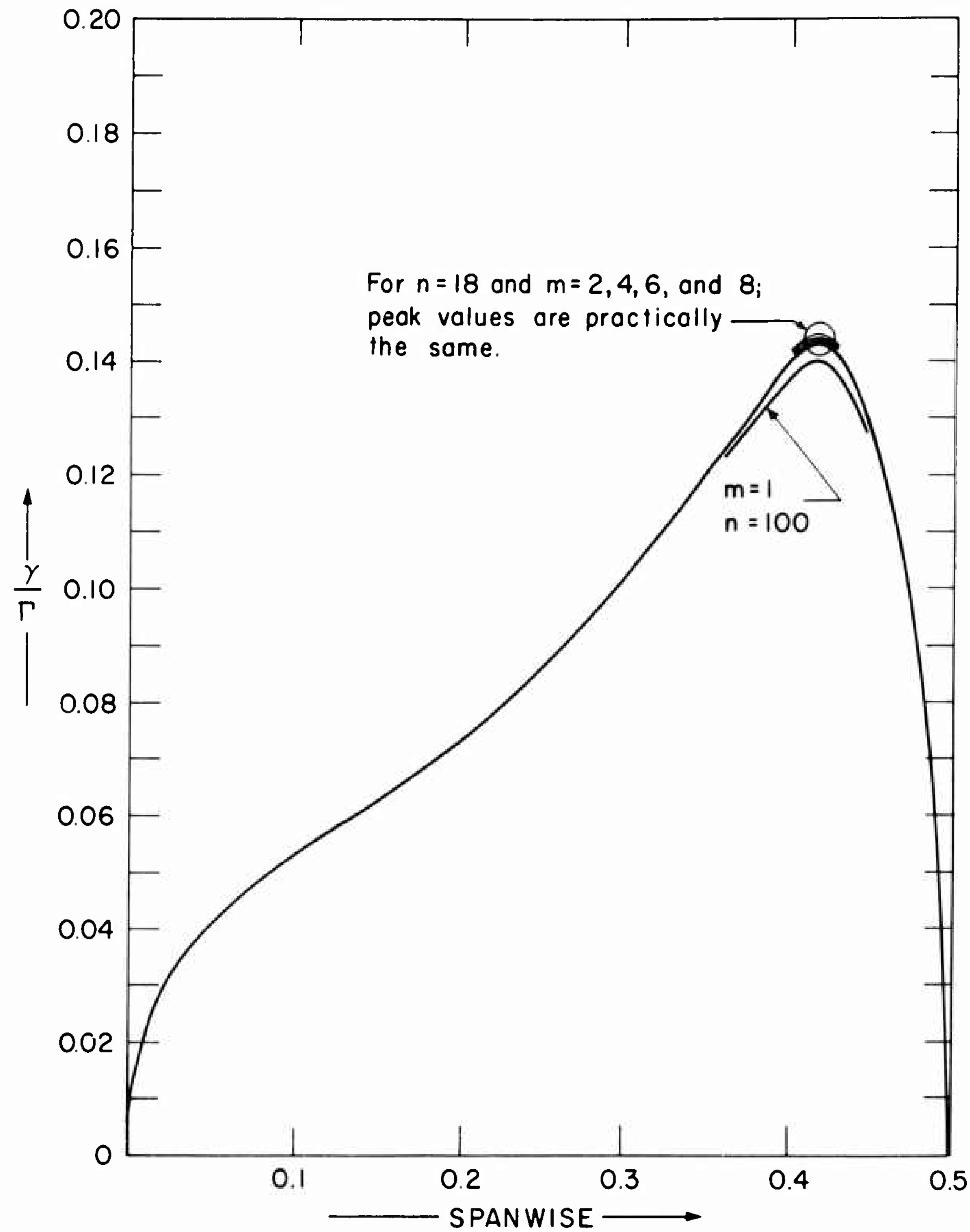


Fig. 8 Spanwise Circulation Distribution For Five Different Values Of d_y.
Z=1 chord, AR=20, ϕ=0, L=0.5

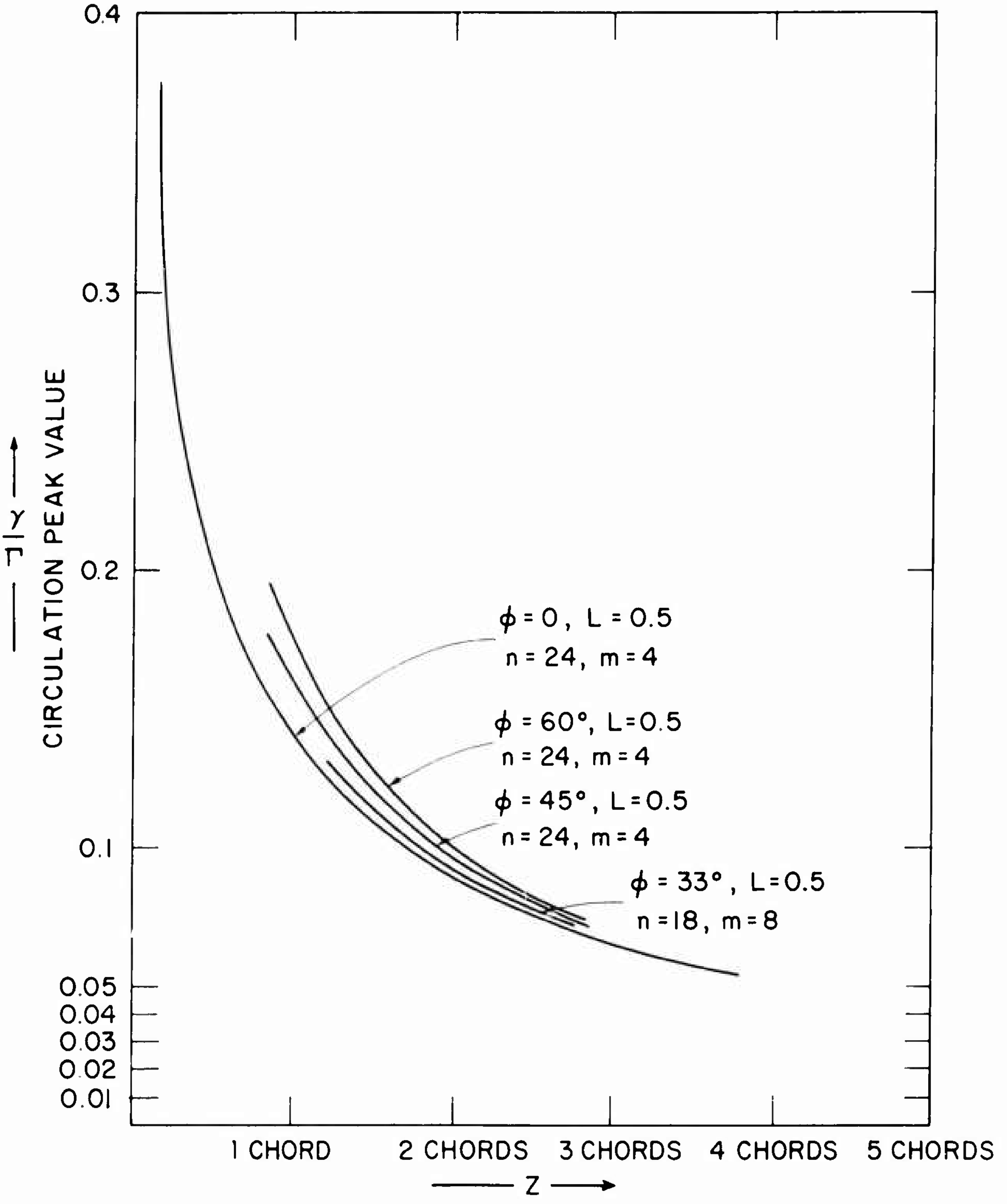


Fig. 9 Convergence Of The Circulation Peak Value As Z Increases

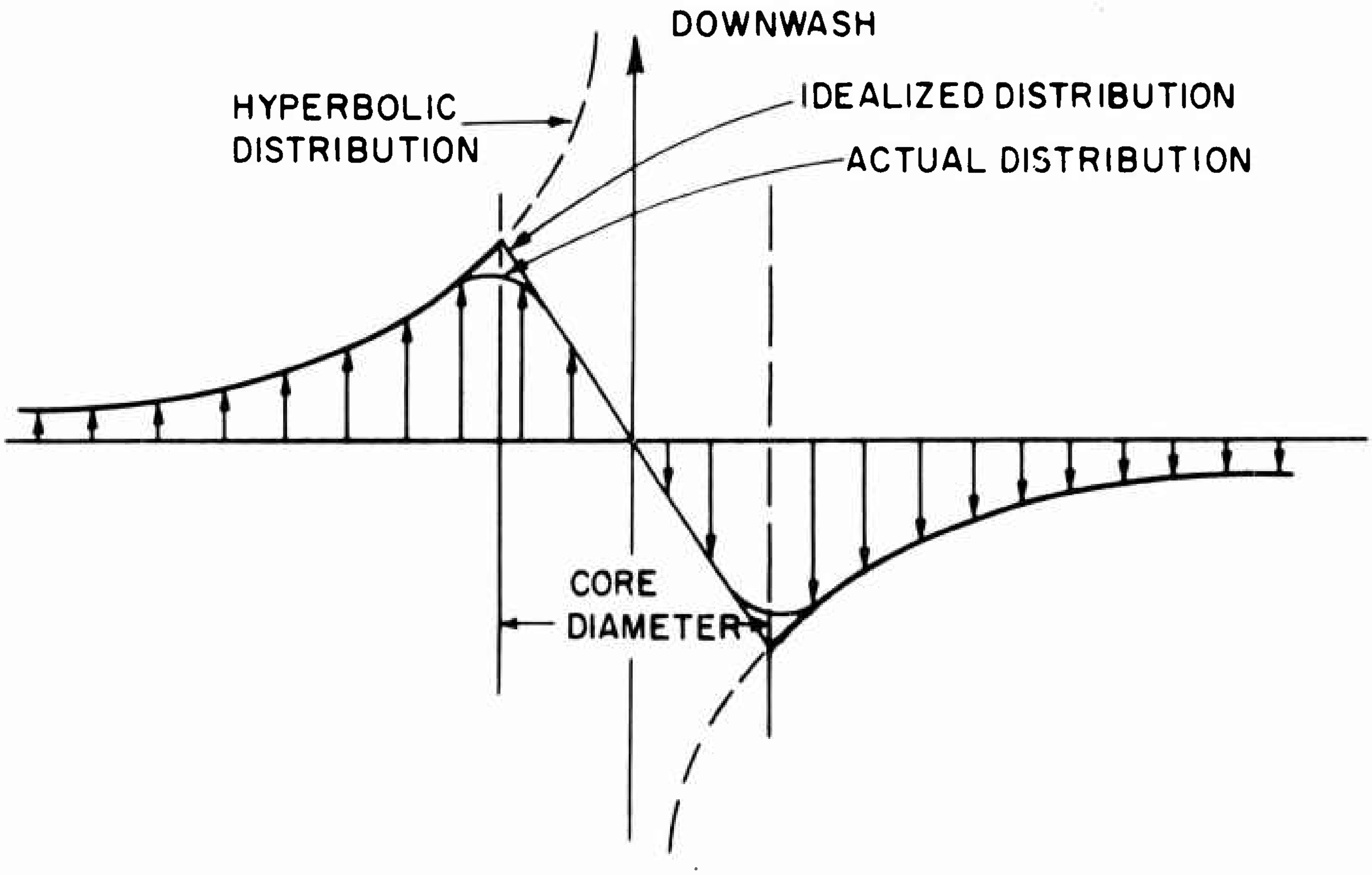


Fig. 10 Velocity Field Due To A Vortex
With A Finite Core Diameter

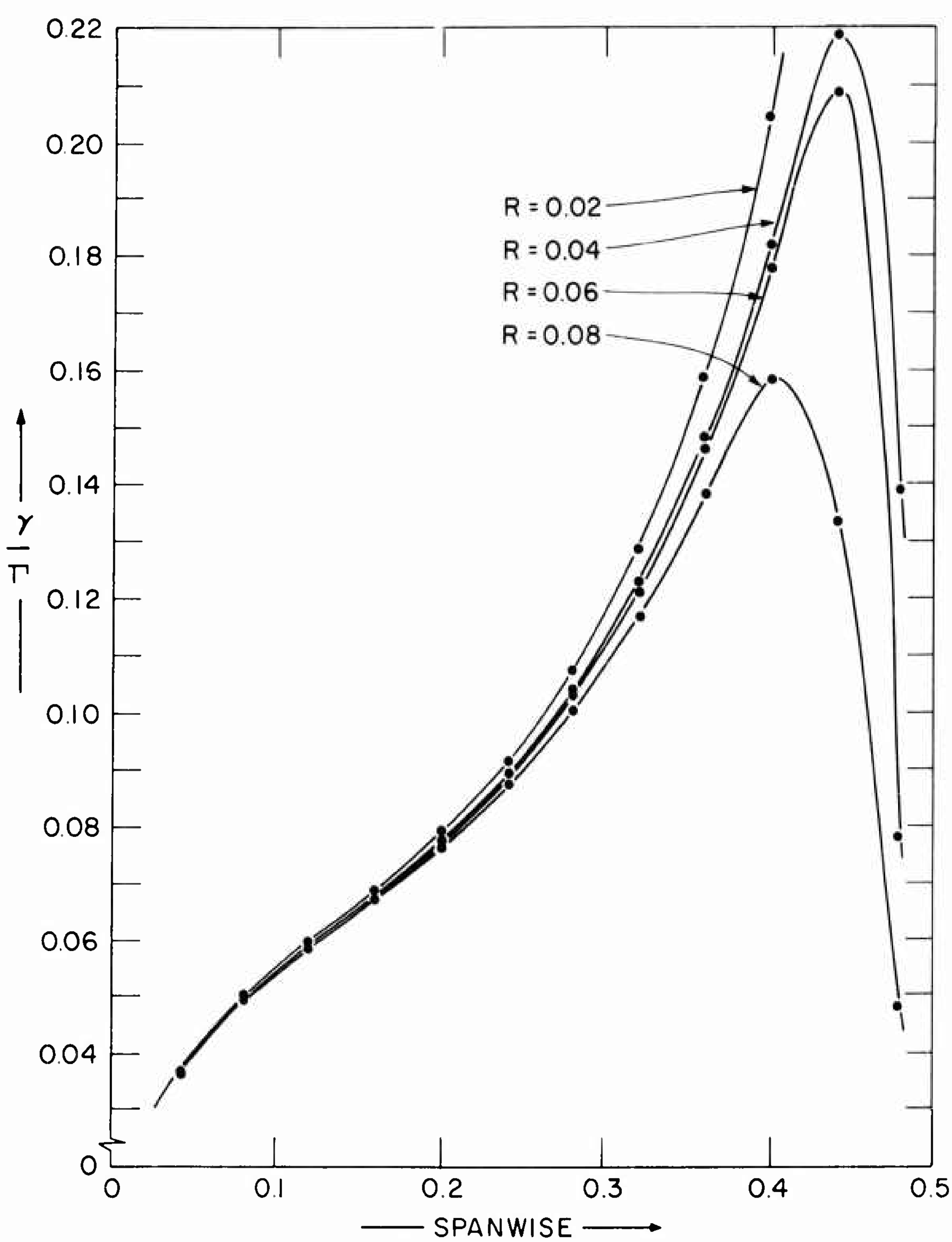


Fig. 11 Spanwise Circulation Distribution For Four Different Core Radii Of The Incident Vortex $R=\frac{\text{Vortex Core Radius}}{\text{Span}}$, Z=0, AR=2, m=4, n=24, Φ=0, L=0.5

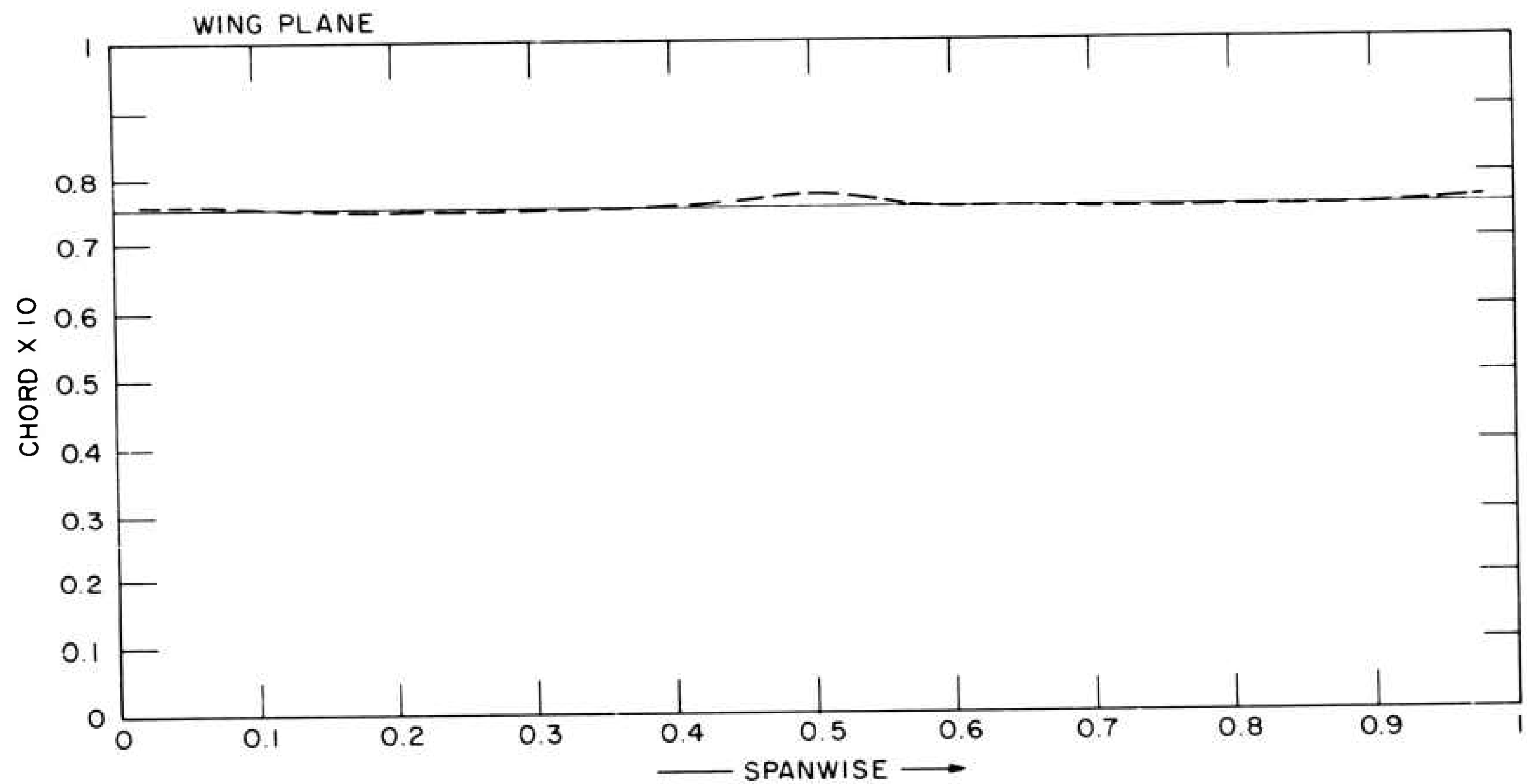


Fig. 12 Locus Of Center Of Pressure
$Z=1$ chord, $AR=20$, $m=4$, $n=18$, $\phi=0$, $L=0.5$

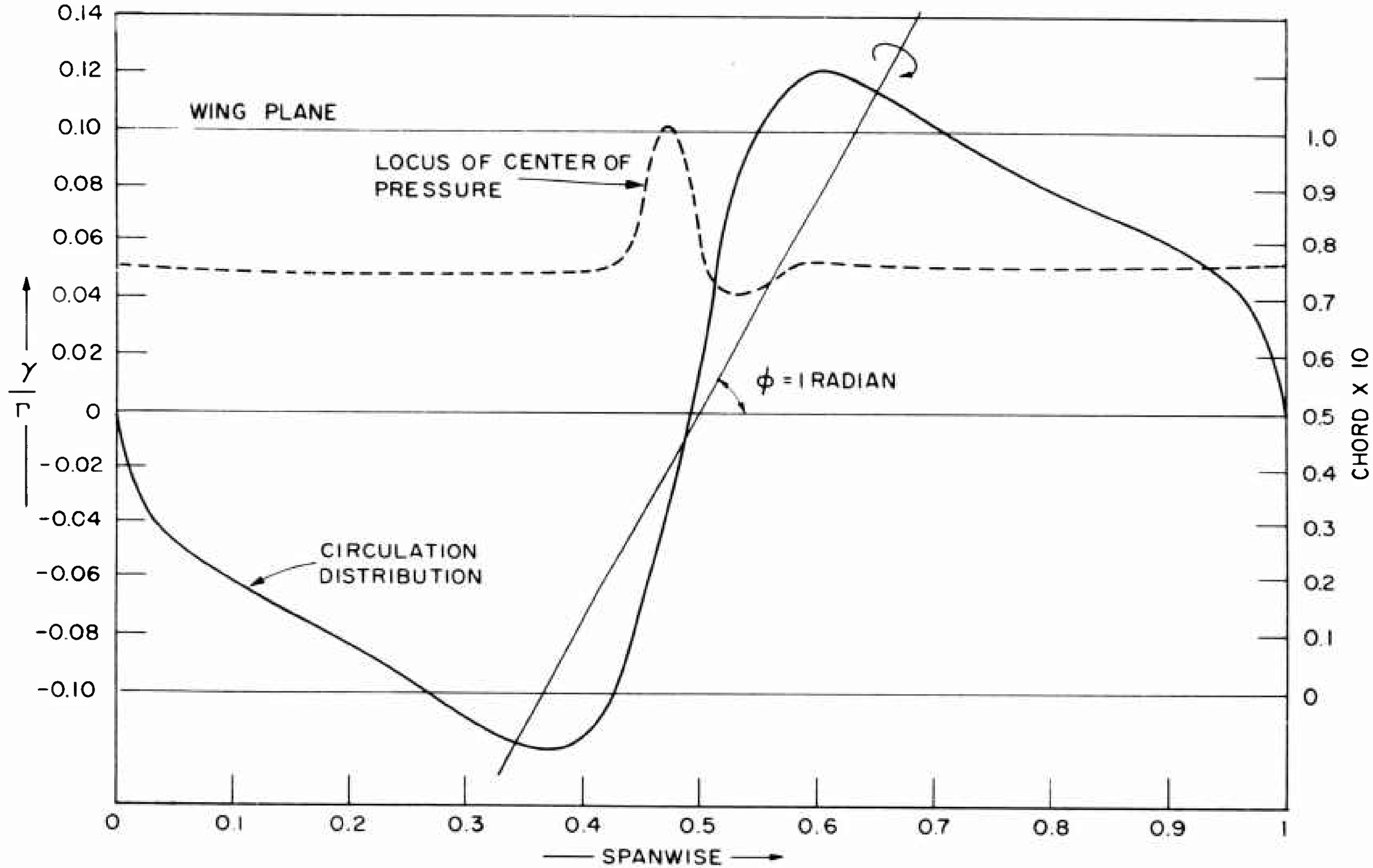


Fig. 13 Spanwise Circulation Distribution And Center Of Pressure For An Incident Vortex Inclined
Z=1.4 chord, AR=20, m=8, n=18

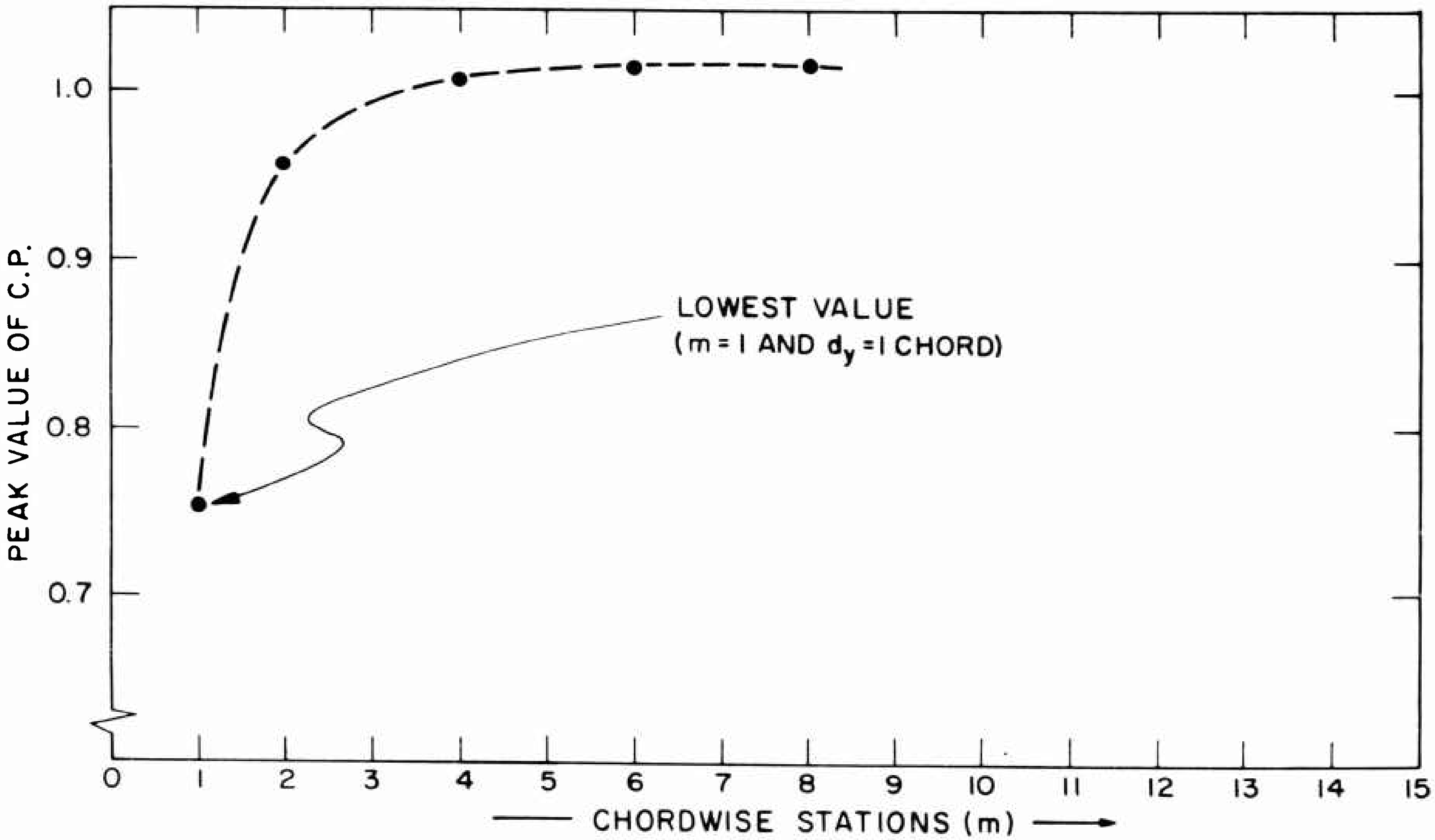


Fig. 14 Convergence of Peak Value of Center of Pressure as d_y Decreases
Z=1 chord, AR=20, m=18, $\phi=33^{o}$, L=0.5

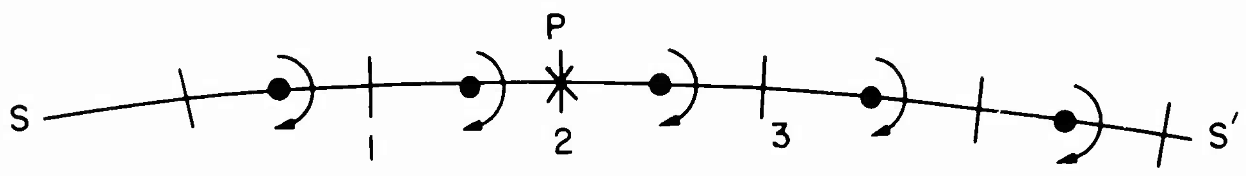


Fig. 15 A One-Dimensional Vortex Sheet Represented By Discrete Vortex Lines

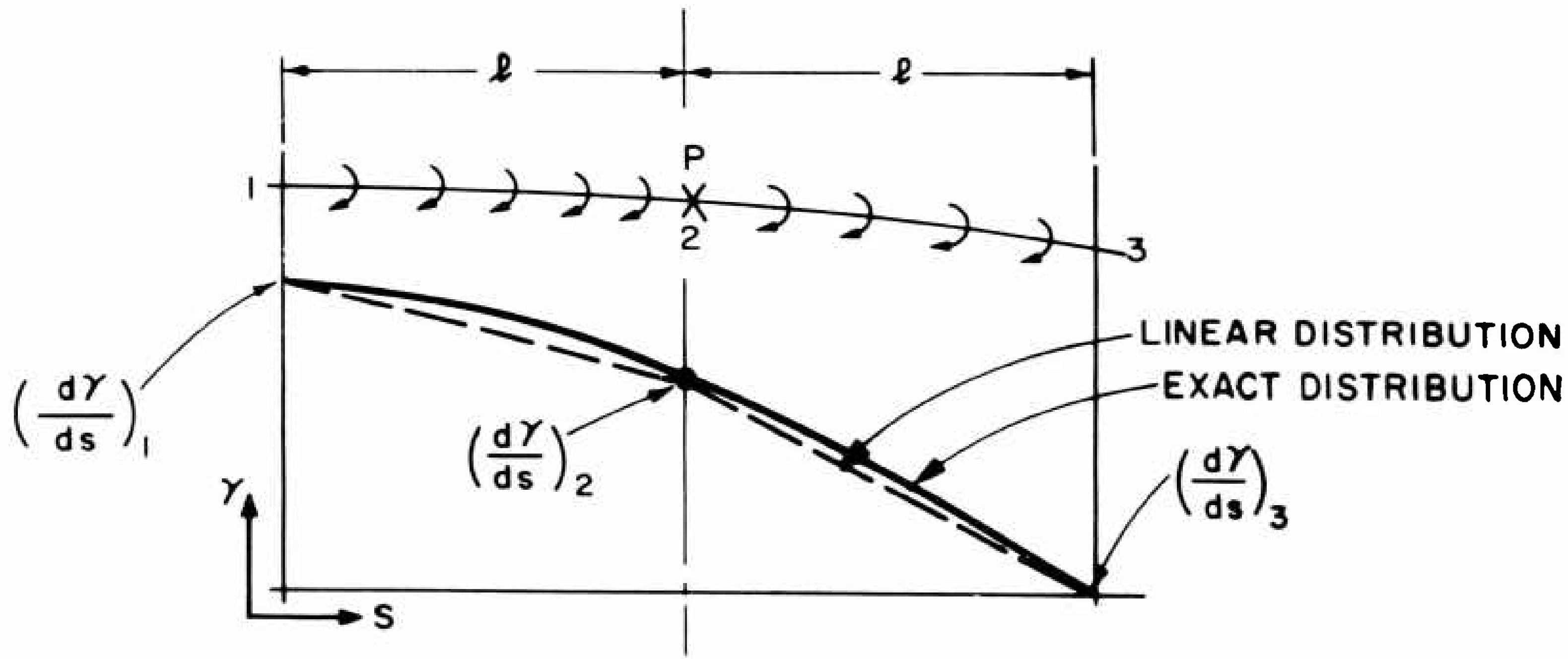


Fig. 16 Linear Approximation To The Circulation Distribution

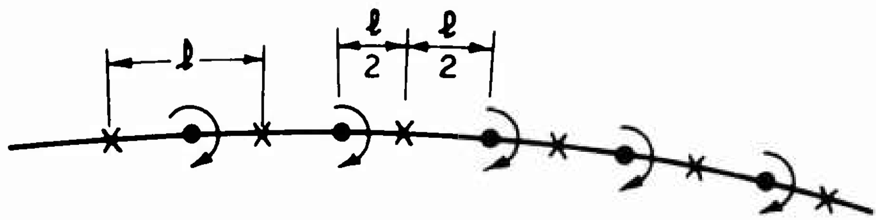


Fig. 17 The Correct Configuration Of The Pivotal Points And The Vortex Lines

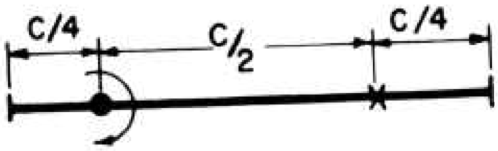


Fig. 18 "Lifting Line" Representation Of A Flat Plate

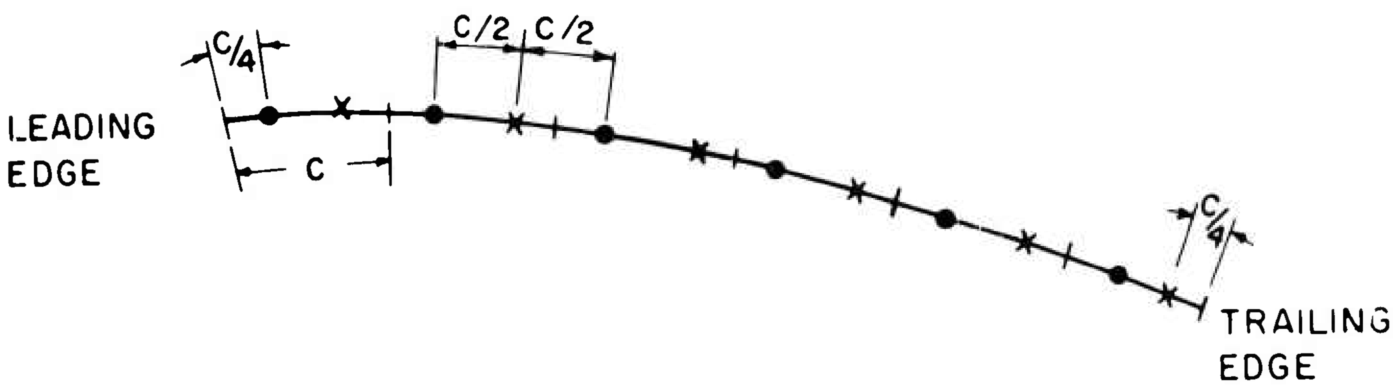
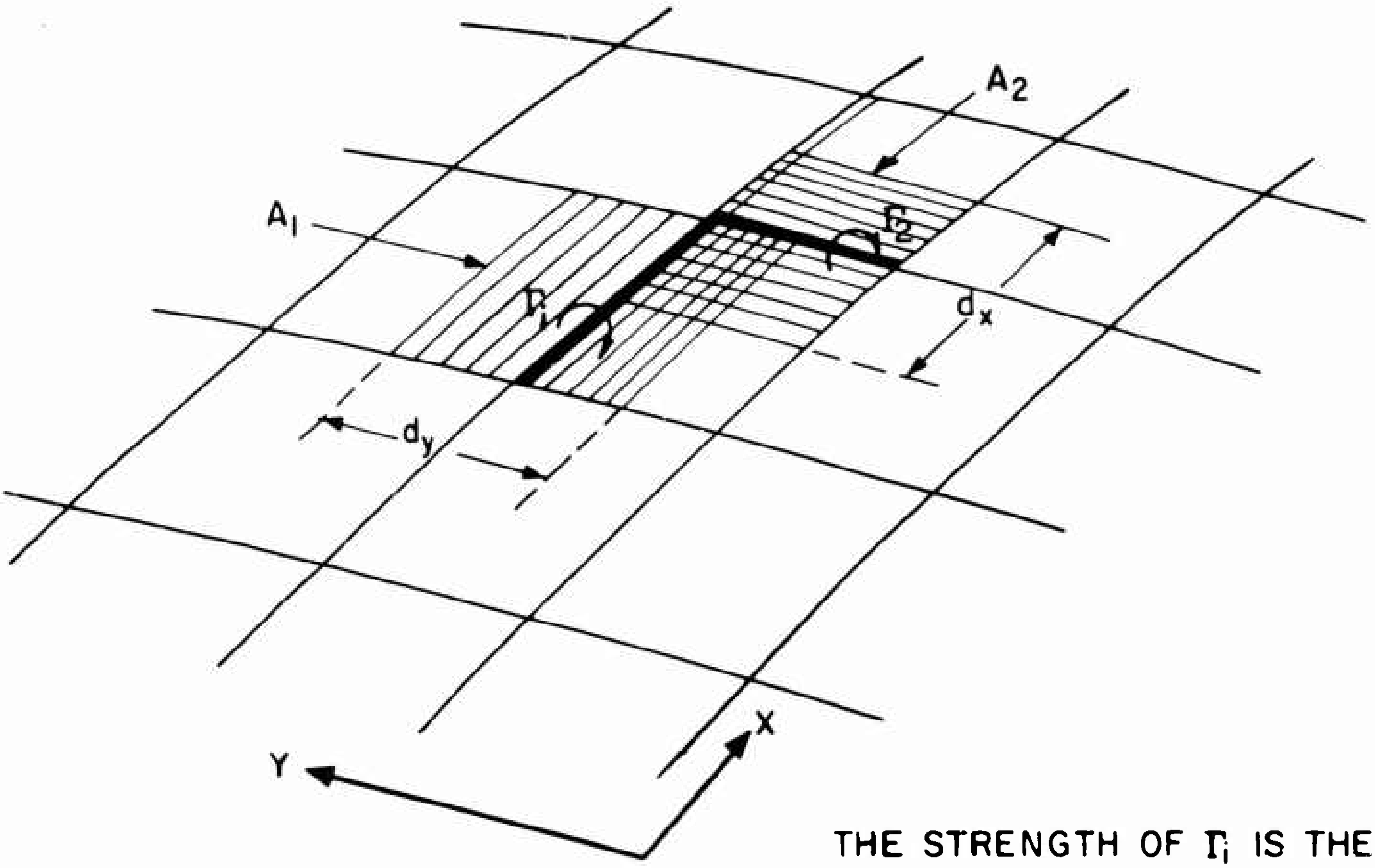


Fig. 19 A "Lifting Surface" Approximated By The Juxtaposition Of Several Flat Plates



THE STRENGTH OF Γ_1 IS THE AVERAGE CIRCULATION OVER AREA A_1 DUE TO THE VORTICITY IN THE X-DIRECTION.

THE STRENGTH OF Γ_2 IS THE AVERAGE CIRCULATION OVER A_2 DUE TO THE VORTICITY IN THE Y-DIRECTION

Fig. 20 Discrete Vortex Line Representation Of A Vortex Sheet

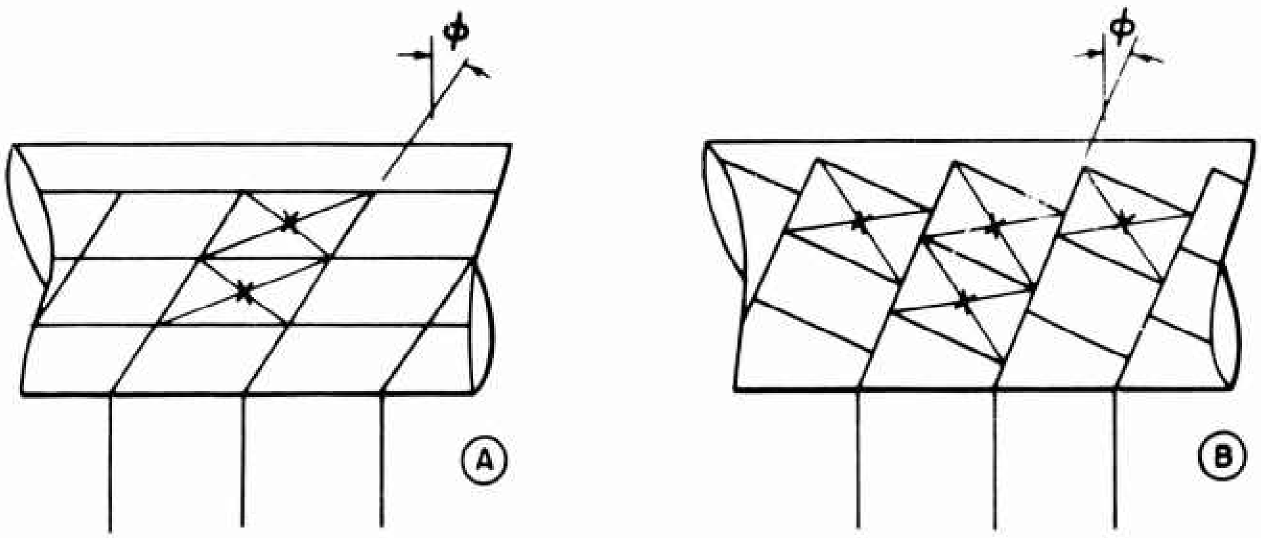


Fig. 21 Tilted Vortex Lattices

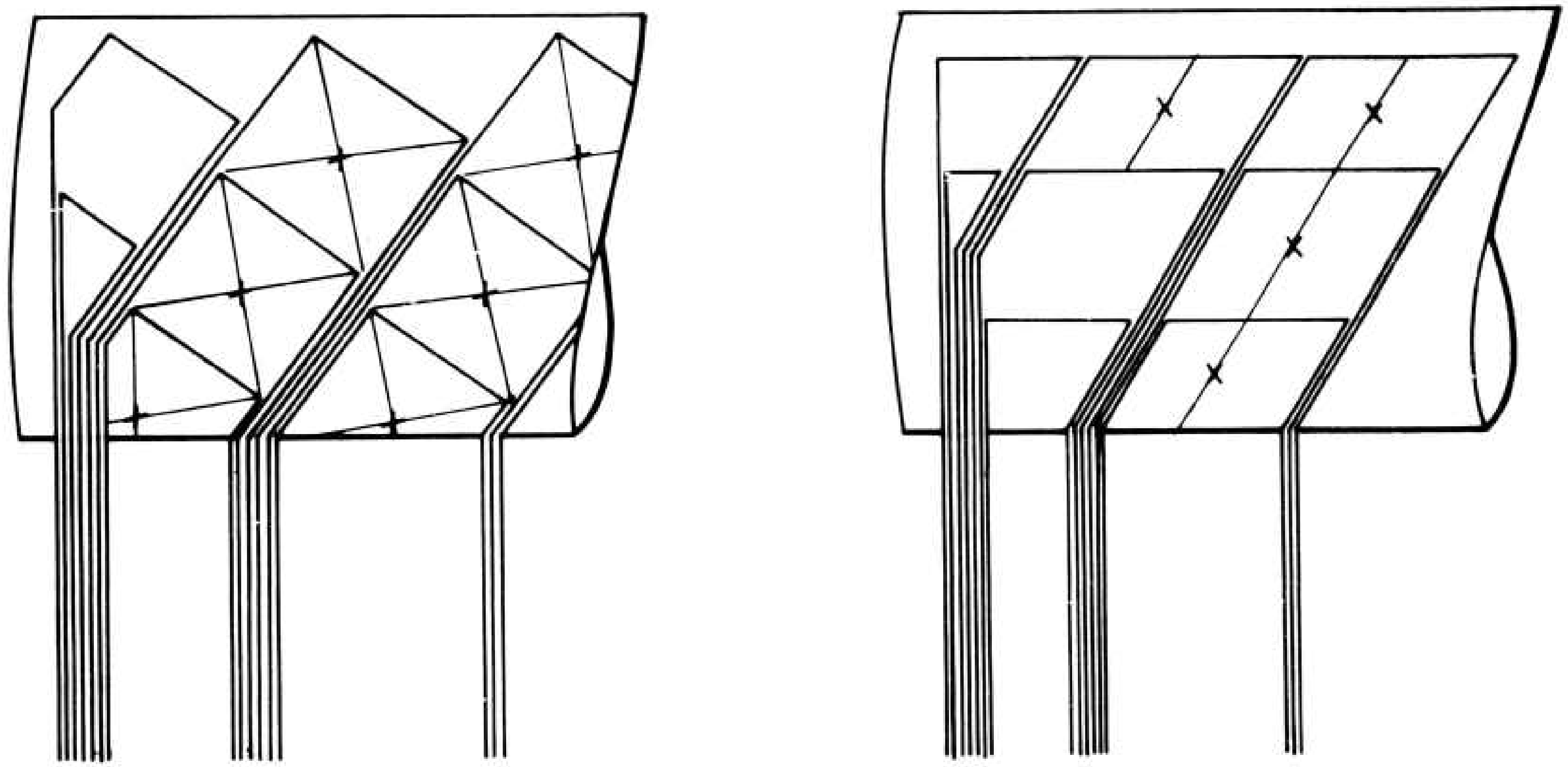


Fig. 22 Tilted Lattice Configurations Near A Wing Tip

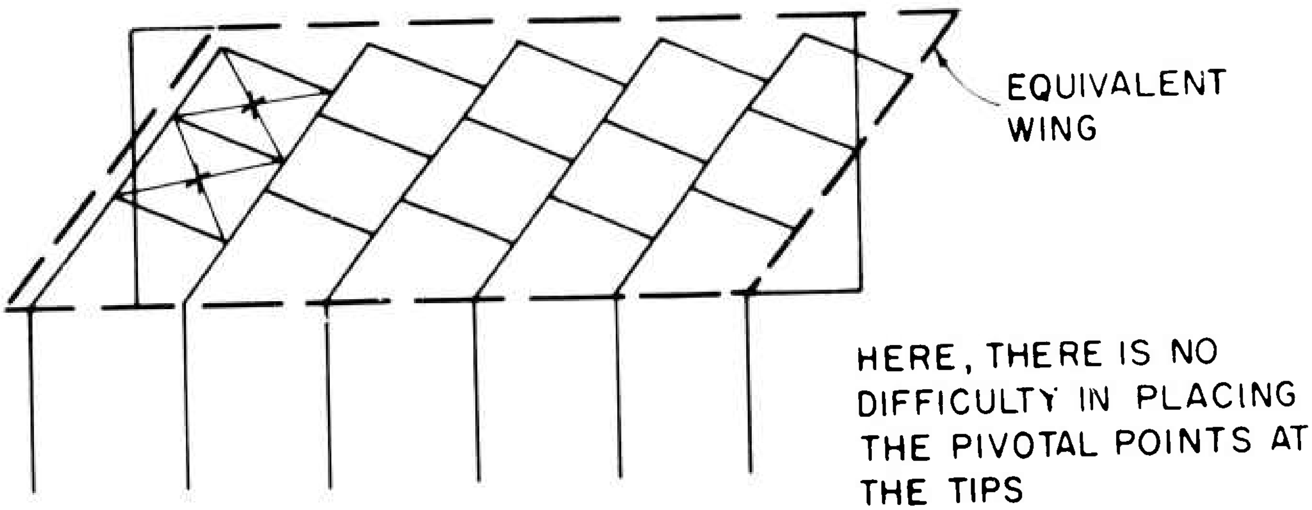


Fig. 23 Use Of An Equivalent Wing For A Tilted Lattice